Math 508 Solutions to Quiz #2 (6/9/03)

1. (a) Find a basis for the span$S$, where S consists of the following six matrices:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(b) Find the linear combinations for the rest of matrices in S in terms of the basis found.

Solution. We proceed the solution in the following steps.

Step 1. Write the pendent equation

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 + x_5A_5 + x_6A_6 = 0. \tag{1}$$

Step 2. From the pendent equation (1) to derive a system of linear equations $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Step 3. Use the Gauss-Jordan elimination to reduce A into its reduced row echelon form

E:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix} \quad (-R_1 + R_2 \to R_2, \quad -R_1 + R_3 \to R_3)$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 1 & -2 & -2 \\ 0 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix} \quad (R_2 \longleftrightarrow R_4)$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 1 & -2 & -2 \end{bmatrix} \quad (-R_2 + R_1 \to R_1, \quad -R_3 \to R_3)$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & -2 & 1 & -2 & -2 \end{bmatrix} \quad (-R_3 + R_2 \to R_2, \quad 2R_2 + R_4 \to R_4)$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & -2 & 2 \end{bmatrix} \quad (R_4 + R_1 \to R_1, \quad -R_4 + R_3 \to R_3, \quad -R_4 \to R_4)$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 \end{bmatrix} =: E$$

Step 4. It follows from E that x_1, x_2, x_3 and x_4 are pivot variables for $Ex = 0$ and $Ax = 0$, and x_5 and x_6 are free variables. Then, going back to the pendent equation (1), we have that A_1, A_2, A_3 and A_4 form a basis for the span$S$.

Step 5. Solve $Ex = 0$. The general solution for $Ex = 0$ is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_5 \begin{bmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \tag{2}$$

where x_5 and x_6 are arbitrary scalars.

- Setting the free variables $x_5 = 1$ and $x_6 = 0$ in (2), we get a particular solution $x = [1, 0, -2, -2, 1, 0]^t$ for $Ex = 0$, which is also a solution to (1). Putting this solution in (1) yields

$$A_5 = -A_1 + 2A_3 + 2A_4. \quad \Big(= [A_1, A_2, A_3, A_4]^t \cdot \text{(5th column of E)} \Big)$$

- Setting the free variables $x_5 = 0$ and $x_6 = 1$ in (2), we get a particular solution $x = [-2, 1, 0, 2, 0, 1]^t$ for $Ex = 0$, which is also a solution to (1). Putting this solution in (1) yields

$$A_6 = 2A_1 - A_2 - 2A_4. \quad \Big(= [A_1, A_2, A_3, A_4]^t \cdot \text{(6th column of E)} \Big)$$