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Solutions to Midterm Test June 30, 2003.

**I.** Determine whether the statement is true or false.

1. If two matrices $A$ and $B$ satisfy $AB = 0$, then either $A = 0$ or $B = 0$.

   **Solution:** False. Counterexample: $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$.

2. If two matrices $A$ and $B$ satisfy $AB = 0$ and $A$ is nonsingular, then $B = 0$.

   **Solution:** True. Since $AB = 0$, it follows that $B = A^{-1}(AB) = A^{-1}0 = 0$.

3. Let $A = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$, where both $I$'s are $n \times n$ identity matrix. Then $A^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix}$.

   **Solution:** True. Verify directly that $\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$.

4. The solution space of the system $\begin{cases} x_1 - x_2 + x_3 + x_4 = 0, \\ 2x_1 + x_2 + 3x_3 - x_4 = 0, \end{cases}$ is equal to the span$\{v_1, v_2\}$, where $v_1 = (-\frac{4}{3}, -\frac{1}{3}, 1, 0)^t$ and $v_2 = (0, 1, 0, 1)^t$.

   **Solution:** True. By the Gauss-Jordan elimination one sees that the general solutions of the systems are given by: $(x_1, x_2, x_3, x_4)^t = \alpha v_1 + \beta v_2$, $\alpha, \beta \in R$.

5. The subset $W$ of $R^{3\times 3}$ that consists of all $3 \times 3$ nonsingular matrices is a subspace of $R^{3\times 3}$.

   **Solution:** False. Note that the zero matrix is not in $W$.

6. The column vectors of any nonsingular matrix in $R^{n\times n}$ form a basis of $R^n$.

   **Solution:** True. The column vectors of any nonsingular matrix are linearly independent.

7. Every nonempty subset of a linearly independent set is linearly independent.

   **Solution:** True. Note that if a subset is linearly dependent, then the whole set is linearly dependent.

8. Given $A \in R^{n\times n}$, $b \in R^n$, then the solution space of $Ax = b$ is a subspace of $R^n$ if and only if $b = 0$.

   **Solution:** True. For if $b \neq 0$, then $x = 0$ is not in the solution space.

9. Let $A \in R^{m\times n}$. Then $Ax = b$ is solvable if and only if $b$ is in the subspace of $R^m$ spanned by the column vectors of $A$.

   **Solution:** True. Note that $Ax = b$ is solvable if and only if there exists at least one $x = (x_1, \cdots, x_n)^t \in R^n$ such that $Ax = b$, i.e.,

   $$b = (a_1, a_2, \cdots, a_n)(x_1, x_2, \cdots, x_n)^t = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n,$$

   i.e., $b \in span\{a_1, a_2, \cdots, a_n\}$, where $a_i (i = 1, \cdots, n)$ are the column vectors of $A$.

10. The column vectors of a matrix corresponding to the pivot columns in its reduced echelon matrix form a basis for the subspace spanned by the column vectors of $A$.

    **Solution:** True.

11. An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ distinct eigenvalues.

    **Solution:** False. The identity matrix is a diagonal matrix with $\lambda = 1$ as its only eigenvalue (with the multiplicity $n$).

12. If $A$ and $B$ have the same eigenvalues, then $A$ and $B$ are similar.

    **Solution:** False. The $2 \times 2$ identity matrix and the Jordan block $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are not similar but their eigenvalues are all equal to 1.

13. If $A$ and $B$ are similar, then their eigenvectors are the same.

    **Solution:** False. Let $P^{-1}AP = B$ for some $P$, i.e, $A = PBP^{-1}$. If $(\lambda, x)$ is an eigenpair of $A$, i. e., $Ax = \lambda x$, then $PBP^{-1}x = \lambda x$. Multiplying both sides (to the left) by $P^{-1}$ yields $B(P^{-1}x) = \lambda(P^{-1}x)$, which says that $(\lambda, P^{-1}x)$ is an eigenpair of $B$. Hence, $A$ and $B$ have same eigenvalues with the different eigenvectors.

14. If the Jordan matrix $J$ for $A$ is

    $$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

    then the dimension of eigenspace for $\lambda$ is 2.

    **Solution:** True. This is because $J$ consists of two Jordan blocks associated with the eigenvalue $\lambda$ of $A$.

15. The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are real.

    **Solution:** False. A simple computation shows that the eigenvalues of $A$ are $\pm i$, where $i$ is the unit pure imaginary number.

16. A nonsingular $n \times n$ matrix $A$ be a nonsingular is diagonalizable if and only if its inverse is diagonalizable.

    **Solution:** True. $A$ is diagonalizable if and only if $P^{-1}AP = D$ for some nonsingular matrix $P$ and a diagonal matrix $D$ if and only if $P^{-1}A^{-1}P = D^{-1}$.

17. If $A$ is singular, then 0 is an eigenvalue of $A$.

    **Solution:** True. The singularity of $A$ implies that $det(A) = 0$ which says that $\lambda = 0$ is the root of the characteristic polynomial $p(\lambda) = det(A - \lambda I)$.

18. The square matrix $A$ and its reduced row echelon matrix $E$ are similar.

    **Solution:** False. Every nonsingular matrix has the identity matrix as its reduced row echelon matrix, while nonsingular matrices can have very different eigenvalues so that they are not necessarily similar.

19. if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

    **Solution:** True. Let $P^{-1}AP = B$ and $Q^{-1}BQ = C$. Then $(PQ)^{-1}A(PQ) = Q^{-1}(P^{-1}AP)Q = Q^{-1}BQ = C$, thereby $A$ and $C$ are similar.

20. There is a unique quadratic which passes through the points $(0,0)$, $(1,1)$, $(2,0)$.

    **Solution:** True. Let $p(t) = at^2 + bt + c$ be the quadratic so that its graph passes through $(0,0)$, $(1,1)$, $(2,0)$. Then $(a,b,c)^t = A^{-1} \cdot (0,1,0)^t$, where $A$ is the Vandermonde matrix with $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$.

**II.** Answer the following:

1. Suppose that $Ap_1 = \lambda p_1$ and $Ap_2 = \lambda p_2 + p_1$. Show that

$$p_2 \in N\Big((A - \lambda I)^2\Big).$$

   **Proof:** It follows from the conditions that $(A-\lambda I)p_1 = 0$ and $(A-\lambda I)p_2 = p_1$. Therefore, $(A - \lambda I)^2 p_2 = (A - \lambda I)p_1 = 0$, i.e., $p_2 \in N\Big((A - \lambda I)^2\Big)$.

2. Assume that a $3 \times 3$ matrix $A$ has eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Write down all the possible Jordan forms for $A$ (except for the order of the Jordan blocks).

   **Solution:** They are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

3. Let $A_k = \begin{bmatrix} \frac{1}{2^k} & \frac{(-1)^k}{3^k} \\ 0 & 0 \end{bmatrix}$ for $k = 1, 2, \cdots$. Find $A_1 + A_2 + \cdots$. (Recall that $1 + r + r^2 + \cdots = \frac{1}{1-r}$ for any $r$, $|r| < 1$.)

   **Solution:** Note that $S(r) := r + r^2 + \cdots = 1/(1 - r) - 1 = r/(1 - r)$. We have $S(1/2) = 1$ and $S(-1/3) = -1/4$. Hence, $A_1 + A_2 + \cdots = \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $B = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$. Find a nonsingular matrix $P$ such that $P^{-1}AP = B$.

   **Solution:** Let $P = (p_1, p_2)$, where $p_1$ and $p_2$ are 2-d column vectors, such that $(Ap_1, Ap_2) = A(p_1, p_2) = AP = PB = (\lambda_2 p_1, \lambda p_2)$. Then $Ap_1 = \lambda_2 p_1$ and $Ap_2 = \lambda_1 p_2$. That is, $(A - \lambda_2 I)p_1 = 0$ and $(A - \lambda_1 I)p_2 = 0$. Hence, $p_1 = \alpha(0,1)^t$ and $p_2 = \beta(1,0)^t$ where $\alpha$ and $\beta$ are arbitrary scalars. Take $\alpha = \beta = 1$ to yield that $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P^{-1} = P$. One can verify directly that $P^{-1}AP = B$.

5. (a) Show that the set $W = \{at^3 + bt^2 + ct + a + b + c : a, b, c \in R\}$ is a subspace of $P_3$; (b) Find a basis for $W$.

**(a) Proof:** Given $p_i(t) = a_i t^3 + b_i t^2 + c_i t + a_i + b_i + c_i \in W$ $(i = 1, 2)$, we have $p_1(t)+p_2(t) = (a_1+a_2)t^3+(b_1+b_2)t^2+(c_1+c_2)t+(a_1+a_2)+(b_1+b_2)+(c_1+c_2) \in W$, and so $W$ is close under the addition operation (defined in $P_3$).

Given $p(t) = at^3 + bt^2 + ct + a + b + c$ and a scalar $\alpha$, we have $\alpha p(t) = (\alpha a)t^3 + (\alpha b)t^2 + (\alpha c)t + \alpha a + \alpha b + \alpha c \in W$, and so $W$ is closed under the scalar multiplication (defined in $P_3$).

Therefore, by the definition we see that $W$ is a subspace of the vector space $P_3$.

**(b) Solution:** Since $p(t) = at^3+bt^2+ct+a+b+c = a(t^3+1)+b(t^2+1)+c(t+1)$, it follows that $W = span\{t^3 + 1, t^2 + 1, t + 1\}$. It is easy to verify that $t^3 + 1$, $t^2 + 1$ and $t + 1$ are linearly independent in $P_3$ and therefore they form a basis for $W$. It follows also that $dim(W) = 3$.

**III.** Find $P$ and $J$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

**Solution:** First find the eigenvalues of $A$. We have $|A - \lambda I| = (-1)^{3+3}(2 - \lambda)det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = (2 - \lambda)[(1 - \lambda)^2 - 1] = (2 - \lambda)(2 - \lambda)(-\lambda)$, and we see that $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 2$ are the eigenvalues of $A$.
Next find the eigenvectors associated with $\lambda_1 = 0$ and $\lambda_2 = 2$. That is, we have to solve the linear systems $Ax = 0$ and $(A-2I)x = 0$ respectively. By the Gauss-Jordan elimination, we have the reduced row echelon matrices for $A$ and $A - 2I$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively. Therefore, the eigenvectors for $\lambda_1$ are $\alpha_1(-1, -1, 1)^t$ and the eigenvectors for $\lambda_2$ are $\alpha_2(-1, 1, 0)^t + \alpha_3(0, 0, 1)^t$, where $\alpha_i$ $(i = 1, 2, 3)$ are any nonzero numbers. Hence, let

$$P = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We have $P^{-1}AP = J$.

**IV.** Find $P$ and $J$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

**Solution:** The characteristic polynomial is the same as in the above problem and so the eigenvalues of $A$ are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 2$. In order to find eigenvectors we note that by the Gauss-Jordan elimination, the reduced row echelon matrices for $A$ and $A - 2I$ are:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively. Therefore, the eigenvectors for $\lambda_1$ are $\alpha_1(1,1,0)^t$ and the eigenvectors for $\lambda_2$ are $\alpha_2(0,0,1)^t$, where $\alpha_i (i = 1, 2)$ are any nonzero numbers. Take $p_1 = (1,1,0)^t$ and $p_2 = (0,0,1)^t$. We have to find a generalized eigenvector $p_3$ of the order 2 for the eigenvalue $\lambda_2 = 2$. Thus we solve $(A - 2I)x = p_2$. By the Gauss-Jordan elimination we have

$$[A - 2I \mid p_2] \longrightarrow \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{2} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and so the general solution for $(A - 2I)x = p_2$ is given by $x = (1/2, -1/2, 0)^t + x_3(0,0,1)^t$. Set $x_3 = 0$ and $p_3 = (1/2, -1/2, 0)^t$. We have

$$P = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

It is easily verified that $P$ is nonsingular and $P^{-1}AP = J$.

**V.** Find $A^{10}$ and $e^{At}$ for the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

**Solution:** Note that $A = 4I + N$ is already a standard Jordan matrix with only one Jordan block. Since $I$ and $N$ commute and $N^k = 0$ for any positive integer $k \geq 3$, we have, by the binomial formula,

$$A^{10} = (4I + N)^{10} = 4^{10}I + 10 \cdot 4^9 N + \frac{10 \cdot 9}{2} \cdot 4^8 N^2 = 4^8 \begin{bmatrix} 16 & 40 & 45 \\ 0 & 16 & 40 \\ 0 & 0 & 16 \end{bmatrix},$$

and, by the Taylor series expansion for $e^{Nt}$,

$$e^{At} = e^{4tI + Nt} = e^{4t}e^{Nt} = e^{4t}(I + Nt + \frac{1}{2}t^2N^2) = e^{4t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$