1 (c). Find a unitary matrix U such that $U^H AU$ is upper triangular.

$$A = \begin{bmatrix} -2 & 4 & 2 \\ 1 & -2 & 1 \\ -3 & 6 & -3 \end{bmatrix}.$$

Solution. We use the Schur decomposition process in two steps to find U and T.

Step 1. Find U_1. We first find the eigenvalues of A. The characteristic polynomial of A is $|\lambda I - A| = \lambda^3 + 7\lambda^2 + 12\lambda = \lambda(\lambda+3)(\lambda+4) = 0$ and hence we have $\lambda_1 = 0$, $\lambda_2 = -3$ and $\lambda_3 = -4$.

Next, we find an eigenvector associated with $\lambda_1 = 0$. By the Gauss-Jordan elimination we have

$$A - \lambda_1 I = \begin{bmatrix} -2 & 4 & 2 \\ 1 & -2 & 1 \\ -3 & 6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so $(A - \lambda_1 I)x = 0$ gives $x_1 = 2x_2$ and $x_3 = 0$. Thus, the eigenspace $E(\lambda_1) = span\{u_1\}$ where

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Now, we extend u_1 to an orthogonal basis u_1, u_2, u_3 for R^3. We first find u_2. Since $(u_2, u_1) = 0$, we see that u_2 is a nonzero solution of $u_1^t x = 0$. Since $u_1^t = [2, 1, 0]$, it follows that the general solution $u_1^t x = 0$ is given by

$$x = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We take $u_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Since $(u_3, u_1) =$ and $(u_3, u_2) = 0$, u_3 is a nonzero solution of

$$\begin{bmatrix} u_1^t \\ u_2^t \end{bmatrix} x = 0. \tag{1}$$

By the Gauss-Jordan elimination we have

$$\begin{bmatrix} u_1^t \\ u_2^t \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the solution space of (1) is equal to $span\{u_3\}$, where $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now let

$$U_1 = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right] = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$U^H AU = \begin{bmatrix} 0 & 3 & \sqrt{5} \\ 0 & -4 & 0 \\ 0 & 3\sqrt{5} & -3 \end{bmatrix}.$$

Step 2. Find U_2. Let $A_1 = \begin{bmatrix} -4 & 0 \\ 3\sqrt{5} & -3 \end{bmatrix}$. The eigenvalues of A_1 are $\lambda_2 = -3$ and $\lambda_3 = -4$. The eigenvectors associated with λ_2 is solutions of $(A + 3I)x = 0$. Since

$$A + 3I = \begin{bmatrix} -1 & 0 \\ 3\sqrt{5} & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Clearly, v_1 and v_2 are orthogonal with a unit length. Let $V = [v_1, v_2]$. We have

$$V^H A_1 V = \begin{bmatrix} -3 & 3\sqrt{5} \\ 0 & -4 \end{bmatrix}.$$

Let

$$U_2 = \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad U = U_1U_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}.$$

Then we have

$$U^H AU = T = \begin{bmatrix} 0 & \sqrt{5} & 3 \\ 0 & -3 & 3\sqrt{5} \\ 0 & 0 & -4 \end{bmatrix}.$$