MA 508 **Name:**________________________

Solution to Test 2 Dec. 4, 2006.

1 [25 pts]. Solve the difference equations with the initial value:

$$\begin{cases} x_1(k+1) = 2x_1(k) + x_2(k), \\ x_2(k+1) = x_1(k) + 2x_2(k), \\ x_3(k+1) = x_2(k) + 2x_3(k), \\ x_1(0) = 1, \\ x_2(0) = 2, \\ x_3(0) = 4. \end{cases}$$

Solution. First write the difference equations in vector form $x(k+1) = Ax(k)$ where

$$A := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Then the solution of given initial value problem is given by $x(k) = A^k x(0)$. In order to compute A^k, we need to factor into the form $A = PJP^{-1}$ where J is in the Jordan canonical form. After computations, we find that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Then we find the eigenvectors of A belonging to λ_1, λ_2 and λ_3 are

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad p_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

respectively. Let $P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. Then

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}.$$

Let $P^{-1}x(0) = c$ so that $Pc = x(0)$ and then using the Gaussian eliminations to solve this system and find

$$c = \begin{bmatrix} 3 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Therefore, we have

$$x(k) = PJ^kP^{-1}x(0) = PJ^kc = 3p_1 + \frac{1}{2}2^kp_2 + \frac{3}{2}3^kp_3 = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2^{k-1}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3^{k+1}}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2 [25 pts]. Let $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ where

$$J_1 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find J^{10}.

Solution. We first write $J_1 = 3I + N_1$ and $J_2 = 2I + N_2$ where

$$N_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_1^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_1^3 = 0,$$

and

$$N_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_2^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_2^3 = \begin{bmatrix} 0 & 0 & & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N_2^4 = 0.$$

Then

$$J_1^{10} = (3I+N_1)^{10} = (3I)^{10} + 10(3I)^9 N_1 + \frac{10 \cdot 9}{2}(3I)^8 N_1^2 = \begin{bmatrix} 3^{10} & 10 \cdot 3^9 & 45 \cdot 3^8 \\ 0 & 3^{10} & 10 \cdot 3^9 \\ 0 & 0 & 3^{10} \end{bmatrix},$$

and

$$\begin{aligned} J_2^{10} &= (2I + N_2)^{10} = (2I)^{10} + 10(2I)^9 N_2 + \frac{10 \cdot 9}{2}(2I)^8 N_2^2 + \frac{10 \cdot 9 \cdot 8}{3!}(2I)^7 N_2^3 \\ &= \begin{bmatrix} 2^{10} & 10 \cdot 2^9 & 45 \cdot 2^8 & 120 \cdot 2^7 \\ 0 & 2^{10} & 10 \cdot 2^9 & 45 \cdot 2^8 \\ 0 & 0 & 2^{10} & 10 \cdot 2^9 \\ 0 & 0 & 0 & 2^{10} \end{bmatrix}. \end{aligned}$$

Thus,

$$J^{10} = \begin{bmatrix} J_1^{10} & 0 \\ 0 & J_2^{10} \end{bmatrix} = \begin{bmatrix} 3^{10} & 10 \cdot 3^9 & 45 \cdot 3^8 & 0 & 0 & 0 & 0 \\ 0 & 3^{10} & 10 \cdot 3^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{10} & 10 \cdot 2^9 & 45 \cdot 2^8 & 120 \cdot 2^7 \\ 0 & 0 & 0 & 0 & 2^{10} & 10 \cdot 2^9 & 45 \cdot 2^8 \\ 0 & 0 & 0 & 0 & 0 & 2^{10} & 10 \cdot 2^9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2^{10} \end{bmatrix}.$$

3 [25 pts]. Solve the initial value problem of $x' = Ax$ and $x(0) = x_0$ where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. We note that from the solution 3 (b) of Test 1 we find $A = PJP^{-1}$ where

$$P = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Hence,

$$e^{Jt} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Using the Gaussian eliminations to solve $Pc = x(0)$ one finds that $c = \begin{bmatrix} -\frac{3}{2} \\ -1 \\ \frac{5}{2} \end{bmatrix}$ Thus,

$$x(t) = e^{At}x(0) = Pe^{Jt}P^{-1}x(0) = Pe^{Jt}c = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} -(\frac{3}{2}+t)e^t \\ -e^t \\ \frac{5}{2}e^{3t} \end{bmatrix}$$

$$= -(\frac{3}{2}+t)e^t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - e^t \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \end{bmatrix} + \frac{5}{2}e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

4 [25 pts]. Let $L(x) = Ax$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find the range of L.
(b) Find an orthogonal basis for range L.
(c) Find the closest vector in range L to the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution. (a) We use $R(L)$ to denote the range of L. Note that $R(L)$ is equal to the span of the columns of A. Using Gaussian eliminations one finds that a basis for $R(L)$ consists of the first two columns of A. Thus,

$$R(L) = span\ \{x_1, x_2\} := span \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(b) Use the Gram-Schmidt method to orthogonalize x_1 and x_2. We let $u_1 = x_1$ and

$$u_2 = x_2 - \frac{(x_2, u_1)}{(u_1, u_1)} u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}.$$

Hence, u_1 and u_2 form an orthogonal basis for $R(L)$.

(c) The closet vector to $x := \begin{bmatrix}1\\2\\3\end{bmatrix}$ in $R(L)$ is the orthogonal projection of x in $R(L)$:

$$x_f = \frac{(x, u_1)}{(u_1, u_1)}u_1 + \frac{(x, u_2)}{(u_2, u_2)}u_2 = \frac{2}{3}u_1 + \frac{\frac{16}{3}}{\frac{24}{9}}u_2 = \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\4\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

5 [Bonus 10 pts]. Let $A = [a_{ij}]$ be a 3×3 matrix.
(i) Show that $(Ax, e_i) = (x, A^t e_i)$ for any $x \in R^3$
(ii) Show that $(Ax, y) = (x, A^t y)$ for any $x, y \in R^3$.