MA 508 **Name:**________________________

Solution to Test 1 October 30, 2006.

1 [20 pts]. Let $W = \left\{ A : A \in \mathbb{R}^{2\times 2} \text{ and } \operatorname{trace} A = 0 \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^{2\times 2}$.

(b) Find a basis for W.

Solution. (a) Note that the matrix $A \in W$ is in the form $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Let A_1 and A_2 in W such that $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$. Then $A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{pmatrix}$ so that $A_1 + A_2 \in W$. Let α be any scalar and $A \in W$. Then $\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & -\alpha a \end{pmatrix} \in W$. Therefore, W is a subspace of $\mathbb{R}^{2\times 2}$.

(b) Note that $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, $W =$ *span* S, where

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

It can be easily check that S is a linearly independent set. Hence, S is a basis for W.

2 [20 pts]. Let $L : \mathbb{R}^3 \mapsto \mathbb{R}^4$ by $L(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find a basis for $N(L)$.

(b) Find a basis for the range of L.

Solution. (a) Use the Gaussian eliminations, we have

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $Ax = 0$ has only zero solution and so $N(L) = \{0\}$.

(b) Note that the range of L is equal to the span of the column vectors of A. From (a) we know that the columns of A are linearly independent. Thus, the columns of A yields a basis for the range of L.

3 [40 pts]. For each of the following matrices, find P and J such that $A = PJP^{-1}$, where J is in the Jordan canonical form.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. (a) Let A be the first one of the above matrices. The characteristic equation is $(1-\lambda)^2(3-\lambda) = 0$ so that the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 3$.
Next, find the eigenvectors of A belonging to $\lambda_1 = \lambda_2 = 1$. Note that

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $(A - I)x = 0$ yields two linearly independent solutions $p_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $p_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, which are two linearly independent eigenvectors of A belonging to $\lambda_1 = 1$. To find the eigenvector of A associated to $\lambda_3 = 3$, we solve the system $(A - 3I)x = 0$. Since

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that the eigenvector $p_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, letting $P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$ yields that $A = PJP^{-1}$ where

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b) Let A be the second matrix given in the problem. Again, one finds easily that A has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 3$. Since

$$A - I = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that A has only one linearly independent eigenvector $p_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. In order to find p_2 we need solve $(A - I)x = p_1$. The augmented matrix for this system is

$$\begin{bmatrix} A - I & p_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which yields a particular solution $p_2 = \begin{bmatrix} 1/20 & -1 \end{bmatrix}$. Now we solve $(A - 3I)x = 0$ for an eigenvector of A associated to $\lambda_3 = 3$. Since

$$A - 3I = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

we find $p_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, let $P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. We have $A = PJP^{-1}$ where

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

4 [10 pts]. Let $J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$ where $\lambda_0 \neq 0$. Is J is invertible? If so, find J^{-1}.

Solution. Since $det\ A = \lambda_0^3 \neq 0$, it follows that A is nonsingular. We have

$$J^{-1} = \begin{bmatrix} \frac{1}{\lambda_0} & -\frac{1}{\lambda_0^2} & \frac{1}{\lambda_0^3} \\ 0 & \frac{1}{\lambda_0} & -\frac{1}{\lambda_0^2} \\ 0 & 0 & \frac{1}{\lambda_0} \end{bmatrix}.$$

5 [10 pts]. Assume that $(A - \lambda_0 I)^2 x_0 \neq 0$ and $(A - \lambda_0 I)^3 x_0 = 0$ for some scalar λ_0 and vector x_0. Let $p_1 = (A - \lambda_0 I)^2 x_0$, $p_2 = (A - \lambda_0 I)x_0$ and $p_3 = x_0$. Show that p_1, p_2 and p_3 are linearly independent.

Solution. Consider the pendent equation

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$$

Multiplying both sides by the matrix $(A - \lambda_0)^2$ and using $(A - \lambda_0)^2 p_1 = (A - \lambda_0 I)^4 x_0 = 0$ and $(A - \lambda_0)^2 p_2 = (A - \lambda_0 I)^3 x_0 = 0$ yields $\alpha_3 (A - \lambda_0 I)^2 p_3 = 0$. That is, $\alpha_3 (A - \lambda_0 I)^2 x_0 = 0$. Since $(A - \lambda_0 I)^2 x_0 \neq 0$ by assumption, we must have $\alpha_3 = 0$. Substitute this into the pendent equation to get $\alpha_1 p_1 + \alpha_2 p_2 = 0$. Multiplying this equation by $A - \lambda_0 I$ gives $\alpha_2 (A - \lambda_0 I) p_2 = 0$, i.e., $\alpha_2 (A - \lambda_0 I)^2 x_0 = 0$ which yields $\alpha_2 = 0$. Thus, the pendent equation reduces to $\alpha_1 p_1 = 0$. Since $p_1 \neq 0$, it follows that $\alpha_1 = 0$. This shows that p_1, p_2 and p_3 are linearly independent.