MA 508 **Name:**________________________

Final Test Dec. 11, 2006.

1. Find a basis for each of the following:

(a) span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$;

(b) span $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

2. Let $L(x) = Ax$ where $A = \begin{bmatrix} 2 & -1 & 1 & -4 \\ 1 & -1 & 1 & -2 \\ 3 & -1 & 1 & -6 \\ 1 & 0 & 0 & -2 \end{bmatrix}$.

(i) Find an orthogonal basis for $N(L)$.

(ii) Find the vector in $N(L)$ that is closest to the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

3. Define the transformation $L : \mathbb{R}^{3\times 3} \mapsto \mathbb{R}$ by $L(A) = \textit{trace}\ A$ where $\textit{trace}\ A = a_{11} + a_{22} + a_{33}$.

(i) Show that L is linear.

(ii) Find a basis for $N(L)$.

4. Prove that if $L : V \mapsto W$ is a one to one linear transformation and $x_1, \cdots, x_n$ are linearly independent in V, then $L(x_1), \cdots, L(x_n)$ are linearly independent.

5. Find $x(100)$ where $x(k)$ is the solution of difference equations $x(k+1) = Ax(k)$ with $x(0) = x_0$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

6. Solve the initial value problem of $x'(t) = Ax(t)$ and $x(0) = x_0$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

7. Let V be an inner product space over $\mathbb{R}$ with an inner product $(\cdot, \cdot)$. Let W be a subspace of V. Define $W^{\perp}$ to be the subspace of all $x \in V$ such that $(x, y) = 0$ for all $y \in W$, i.e.,

$$W^{\perp} = \{x \in V : \ (x, y) = 0 \quad \text{for all } y \in W\}.$$

(a) Prove that $W^{\perp}$ is also a subspace of V.

(b) Prove that $W \cap W^{\perp} = \{0\}$.

(c) Let $x_1, \cdots, x_k$ be an orthogonal basis of W. Define $L : V \mapsto V$ by

$$L(x) = \sum_{i=1}^{k} (x, x_i) x_i.$$

Show that (i) L is linear; (ii) the range of L is W; and (iii) the kernel of L is $W^{\perp}$.

(d) (**Bonus**) Assume that $\dim(V) < \infty$. Show that

$$\dim(W) + \dim(W^{\perp}) = \dim(V).$$