MA456/506(UAH) Practice Problems For the Final (April 23, 2008)

1. Solve $xu_x + yu_y = xy$ with $u = \frac{1}{2}x^2$ on $y = x$. (Solution: $u(x,y) = \frac{1}{2}xy + f(\frac{y}{x})$ where f is any function such that $f(1) = 0$.)

2. Find the solution in $x > 0$ of $xu_x + yu_y = u - 2xy$ with $u = 2y^2 + 2$ on $x = 2$. (Solution: $x(1 + \frac{2y}{x})^2 - 2xy$.)

3. Find the general solution of the equation $au_x + bu_y + cu = 0$ where a, b, c are constants and $a \neq 0$. (Solution: $u(x,y) = e^{-\frac{c}{a}x} f(bx - ay)$ where f is any function.)

4. Find the general solution of the equation $u_{xx} - u_{tt} = t + e^{2x}$. (Solution: $u(x,t) = f(x+t) + g(x-t) + t^3/6 - e^{2x}/4$ where f and g are any functions.)

5. Find the general solution of the equation $3u_{xx} - 5u_{xy} - 2u_{yy} + 3u_x + u_y = 2$. (Solution: $u(x,y) = f(2x+y)e^{\frac{1}{8}(3y-x)} + g(3y-x) + \frac{2}{8}(2x+y)$ where f and g are any functions.)

6. Find the solution of the problem on the half line:

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \qquad 0 < x < \infty, \quad t > 0, \\ &u(0,t) = 0, \qquad t > 0, \\ &u(x,0) = \sin^3 x, \quad u_t(x,0) = 0 \qquad 0 < x < \infty \end{aligned}$$

and prove the solution $u(x,t) \in C^2((0,\infty) \times \mathbb{R})$. (Solution: $u(x,t) = \frac{1}{2}(\sin^3(x+t) + \sin^3(x-t))$.)

7. Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = xt, \qquad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = u_t(x,0) = 0 \qquad -\infty < x < \infty. \end{aligned}$$

(Solution: $u(x,t) = \frac{1}{6}xt^3$.)

8. Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = \sin x, \qquad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \cos x, \quad u_t(x,0) = x \qquad -\infty < x < \infty. \end{aligned}$$

(Solution: $u(x,t) = xt + \cos x \cos t + \sin x(1 - \cos t)$.)

9. Solve the problem

$$u_t - u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = e^{-x} \qquad -\infty < x < \infty.$$

(Solution: $u(x,t) = e^{t-x}$.)

10. Solve the problem

$$u_t - u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = e^{-x^2} \qquad -\infty < x < \infty.$$

(Solution: $u(x,t) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}$.)

11. Solve the problem

$$u_t - u_{xx} = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \begin{cases} 2, & \text{if } x > 0, \\ 4, & \text{if } x < 0. \end{cases}$$

(Solution: $u(x,t) = 3 - erf(\frac{x}{2\sqrt{t}})$.)

12. Solve the Cauchy problem for the advection-diffusion equation

$$u_t - ku_{xx} + cu_x = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \phi(x), \quad -\infty < x < \infty.$$

(Hit: Make the change of the independent variable $y = x - ct$ to replace x.)

13. Use the method of separation of variables to find the solution of

$$u_t = u_{xx} - 4u, \qquad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u_x(\pi,t) = 0, \qquad t > 0,$$
$$u(x,0) = x(\pi - x) \qquad 0 < x < \pi.$$

Hint. First transform the problem for the new dependent variable $v(x,t) = u(x,t)e^{-4t}$ and then apply the method of separation of variables. Solution:

$$u(x,t) = 8e^{4t} \sum_{n=1}^{\infty} [\frac{4}{\pi}(2n-1)^{-3} + (-1)^n (2n-1)^{-2}] e^{-(2n-1)^2 t/4} \sin \frac{(2n-1)x}{2}]$$

14. Use the method of separation of variables to find the solution of

$$u_t = u_{xx}, \qquad 0 < x < \pi, \quad t > 0,$$
$$u_x(0,t) = u_x(\pi,t) = 0, \qquad t > 0,$$
$$u(x,0) = \sin x, \qquad 0 < x < \pi.$$

(Solution: $u(x,t) = 2 + 2\sum_{n=1}^{\infty} \frac{[(-1)^{n+1}-1]}{2+n^2} e^{-n^2 t} \cos nx$.)

15. Use the method of separation of variables to find the solution of

$$\begin{aligned} &u_{tt} = u_{xx}, \qquad 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = u_x(1,t) = 0, \qquad t > 0, \\ &u(x,0) = x\cos \pi x, \quad u_t(x,0) = 1, \qquad 0 < x < 1. \end{aligned}$$

(Solution: $u(x,t) = (-\frac{1}{2\pi}\cos \pi t + \frac{4}{\pi^2}\sin \pi t)\sin \pi x + \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n n}{n^2-1}\cos n\pi t + \frac{1-(-1)^n}{\pi n^2}\sin n\pi t\right)\sin n\pi x$.)

16. Use the method of separation of variables to find the solution of

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \qquad 0 < x, y < \pi, \\ &u_y(x,0) = u_y(x,\pi) = 0, \qquad 0 < x < \pi, \\ &u_x(0,y) = 0, \quad u_x(\pi,y) = \cos 3y, \qquad 0 < y < \pi. \end{aligned}$$

(Solution: $u(x,y) = \frac{1}{3\sinh 3\pi}\cosh 3x \cos 3y$.)

17. Use the method of separation of variables to find the solution of

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \qquad 0 < x, y < \pi, \\ &u_y(x,0) + u(x,0) = 0, \quad u(x,\pi) = \sin\frac{3x}{2}, \qquad 0 < x < \pi, \\ &u(0,y) = 0, \quad u_x(\pi,y) = 0, \qquad 0 < y < \pi. \end{aligned}$$

(Solution: $u(x,y) = \frac{3\cosh\frac{3y}{2} - 2\sinh\frac{3y}{2}}{3\cosh\frac{3\pi}{2} - 2\sinh\frac{3\pi}{2}}\sin\frac{3x}{2}$.)