Solutions to Test 2 **Name:** **S. Ai**
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1. Use the definition of the limit (i.e., the $\varepsilon - \delta$ language) to show that

$$\lim_{x\to 5} x^2 - 3x + 1 = 11.$$

Proof. First we have

$$|(x^2 - 3x + 1) - 11| = |x^2 - 3x - 10| = |x + 2||x - 5|.$$

Given an $\epsilon > 0$, we first let $\delta_1 = 1$. Then $|x - 5| < \delta_1$ implies that $|x - 5| < 1$ and so $4 < x < 6$ and so $|x + 2| = x + 2 < 8$. We therefore define $\delta = \min\{\delta_1, \frac{\epsilon}{8}\}$. Then, for $|x - 5| < \delta$, we have $|(x^2 - 3x + 1) - 11| < 8\delta \le \epsilon$. This shows that $\lim_{x\to 5} x^2 - 3x + 1 = 11$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 6x$ if x is rational and $f(x) = x^2 + 8$ if x is irrational. Prove that f is discontinuous at $x = 1$ and continuous at $x = 2$. Are there any other points besides 2 at which f is continuous?

Proof. First show that f is discontinuous at $x = 1$. Take a rational sequence $x_n = 1 + \frac{1}{n}$ and an irrational sequence $y_n = 1 + \frac{\sqrt{2}}{n}$. Clearly, $x_n \to 1$ and $y_n \to 1$ as $n \to \infty$. However, $f(x_n) = 6x_n \to 6$ and $f(y_n) = y_n^2 + 8 \to 9$ as $n \to \infty$. This shows that $\lim_{x\to 1} f(x)$ does not exist. Hence, f is discontinuous at $x = 1$.
Now we show that f is continuous at $x = 2$. Note that $f(2) = 6 \times 2 = 12$ and

$$|f(x) - f(2)| = \begin{cases} 6|x-2|, & \text{if } x \in \mathbb{Q}, \\ |x+2||x-2|, & \text{otherwise.} \end{cases}$$

Hence, given any $\epsilon > 0$, let $\delta = \min\{\frac{\epsilon}{6}, 1, \frac{\epsilon}{5}\} = \min\{\frac{\epsilon}{6}, 1\}$. Then, for $|x - 2| < \delta$, we have

$$|f(x) - f(2)| = \begin{cases} 6\delta \le \epsilon, & \text{if } x \in \mathbb{Q}, \\ 5\delta < \epsilon, & \text{otherwise.} \end{cases}$$

This shows that f is continuous at $x = 2$.
We note that for such a function f, it is continuous only at the intersection points of the two functions $y = 6x$ and $y = x^2 + 8$, which occur at $x = 2$ and $x = 4$. Hence, f is also continuous at $x = 4$.

3. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$ and suppose that $f(c) > 0$. Show that there is a neighborhood U of c such that $f(x) > f(c)/2$ for all $x \in U \cap D$.

Proof. Let $\epsilon = f(c)/2 > 0$. Since f is continuous at $x = c$, it follows that there exists a $\delta > 0$ such that for $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$, which yields $f(x) - f(c) > -\epsilon$ and so $f(x) > f(c) - \epsilon = f(c)/2$.

4. Use the intermediate value theorem to show that the equation $3^x = x^2$ has at least one real solution.

Proof. Let $f(x) = 3^x - x^2$. f is continuous on $(-\infty, \infty)$. Since $f(-1) = \dfrac{1}{3} - 1 = -\dfrac{2}{3}$ and $f(0) = 1$, it follows from the intermediate value theorem that $f(c) = 0$ for some $c \in (-1, 0)$. Thus, $x = c$ gives a real solution to the equation $3^x = x^2$.

5. Use the definition to show that the function $f(x) = x^2 + 3x + 6$ is uniformly continuous on $(0, 4)$.

Proof. First note that for $x, y \in (0, 4)$, $|f(x) - f(y)| = |x + y + 3||x - y| < 11|x - y|$. Hence, given an $\epsilon > 0$, let $\delta = \dfrac{\epsilon}{11}$. Then, for any $x, y \in (0, 4)$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < 11\delta = \epsilon$. This shows that f is uniformly continuous on $(0, 4)$.

6. Let $f(x) = x^3$ for $x \in [0, b]$ where $b > 0$. Let P_n be the partition of $[0, b]$ which divide $[0, b]$ into n subintervals of an equal length, i.e.,

$$P = \{0, \frac{b}{n}, \frac{2b}{n}, \cdots, \frac{(n-1)b}{n}, b\}.$$

(i) For $n \in \mathbb{N}$, find $U(f, P_n)$ and $L(f, P_n)$.
(ii) Use the definitions of $U(f)$ and $L(f)$ show that $U(f) \leq b^4/4$ and $L(f) \geq b^4/4$, so that $\int_0^b f(x)\,dx = U(f) = L(f) = b^4/4$.

7. (**Bonus**) (i) Let $f(t) = t$ for $0 \leq t \leq 2$ and $f(t) = 3$ for $2 < t \leq 4$.
(a) Find an explicit expression for $F(x) = \int_0^x f(t)\,dt$ as a function of x.
(b) Sketch F and determine whether F is differentiable at $x = 2$.

(ii) Do the same for the function $f(t) = t$ for $0 \leq t \leq 2$ and $f(t) = 2$ for $2 < t \leq 4$.