Solutions to Test 1 **Name:** **S. Ai**
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I. Mark each statement True or False. Justify each answer.

1. If a nonempty subset of $\mathbb{R}$ has an upper bound, then it has a least upper bound.

   Solution. True. This is just the Axiom of the completeness for $\mathbb{R}$.

2. Let S be a nonempty subset of $\mathbb{R}$ and $x \notin S$. Then x is a boundary point of S if and only if x is an accumulation point of S.

   Solution. True. Since $x \notin S$, the assertion follows from the definitions for accumulation points and boundary points respectively.

3. Let A be a nonempty open subset of $\mathbb{R}$. Then $A \cap \mathbb{Q} \neq \emptyset$.

   Solution. True. Since A is not empty, there is a $x \in A$. Since A is open, then x is an interior point of A, which implies that there is a neighborhood $(x-\epsilon, x+\epsilon) \subset A$. By the density of rational numbers we have $(x-\epsilon, x+\epsilon) \cap \mathbb{Q} \neq \emptyset$. Hence, $A \cap \mathbb{Q} \neq \emptyset$.

4. Every finite subset of $\mathbb{R}$ is compact.

   Solution. True. Such a set is closed and bounded

5. A set is compact if and only if it has maximum and minimum.

   Solution. False. If a set is compact, then it has maximum and minimum. The converse is false. For example, the set $A = [-1, 1] \setminus \{0\}$, and the set of all rational numbers in $[0, 1]$

6. The intersection of any collection of compact sets is compact.

   Solution. True.

7. Let $A_n = (0, \frac{1}{n})$ and $B_n = [n, \infty)$ for $n \in \mathbb{N}$. Find $\cap_{n=1}^{\infty} A_n$ and $\cap_{n=1}^{\infty} B_n$. Do your results contradict the Nested Intervals Theorem?

   Solution. $\cap_{n=1}^{\infty} A_n = \emptyset$ and $\cap_{n=1}^{\infty} B_n = \emptyset$. The results do not contradicts the nested interval theorem since A_n are not closed and b_n are not bounded so that both are not compact.

8. $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \sqrt{n+1} - \lim_{n\to\infty} \sqrt{n} = \infty - \infty = 0.$

   Solution. False. The difference rule for the limits only applies when each limit exists.

9. An increasing sequence either converges or diverges to ∞.

   Solution. True. If such a sequence (x_n) is bounded, then it converges to the finite number $\sup\{x_n : n \in \mathbb{N}\}$; else, it diverges to ∞.

10. If a sequence is bounded and monotone, then it is Cauchy.

    Solution. True. Such a sequence converges and thererore it is Cauchy.

II. Let S be the set of irrational numbers in $[0,1]$. Construct an open cover for S that does not have any finite subcover for S.

Solution. Let $A_0 = (-1, \frac{1}{2})$ and $A_n = (\frac{1}{2} + \frac{1}{n}, 2)$ for $n \geq 1$. Then, $\cup_{n=1}^{\infty} A_n = (-1,2) \setminus \{\frac{1}{2}\} \supsetneqq S$. Hence, $\{A_n\}$ is an open cover for S. However, given any finite subsets $A_{n_1}, \cdots, A_{n_k}$, we assume that n_k is the largest subindex. We have

$$\cup_{i=1}^{k} A_{n_i} \subset \cup_{n=0}^{n_k} A_n = (-1, \frac{1}{2}) \cup (\frac{1}{2} + \frac{1}{n_k}, 2).$$

Hence, $\cup_{i=1}^{k} A_{n_i}$ does not contain the irrational points in $(\dfrac{1}{2}, \dfrac{1}{2} + \dfrac{1}{n_k}]$, which implies that $A_{n_1}, \cdots, A_{n_k}$ do not form an open cover for S.

III. Use the definition of the limit to show that $\displaystyle\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

Proof. Given an $\epsilon > 0$, let $N = 1/\epsilon$. Then, for $n > N$, we have

$$\frac{1}{n(n+1)} < \frac{1}{n} < \frac{1}{N} = \epsilon.$$

Hence, by the definition of the limit it follows that $\displaystyle\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$.

IV. Let $s_1 = 1$ and $s_{n+1} = 1 + \dfrac{s_n}{1+s_n}$ for $n \in \mathbb{N}$. Use the Monotone Convergence Theorem to show that (s_n) converges as $n \to \infty$ and find its limit.

Proof. First show by the induction method that (s_n) is increasing. Clearly, $s_2 = 1 + \frac{1}{2} > s_1$. Assume that $s_k > s_{k-1}$ for some $k \geq 2$. Note that the function $f(x) = 1 + \dfrac{x}{1+x}$ with $f'(x) = \dfrac{1}{(1+x)^2}$ so that f is increasing on $(-\infty, \infty) \setminus \{-1\}$. Also note that $s_n \geq 1$. It follows that $s_{k+1} = f(s_k) > f(s_{k-1}) = s_k$. This shows that (s_n) is increasing.

Secondly, we show that s_n is bounded above. Since $s_{n+1} > s_n$, we have $1+\dfrac{s_n}{1+s_n} > s_n$. It follows that $s_n - 1 < \dfrac{s_n}{1+s_n}$ and so $s_n^2 - s_n - 1 < 0$, which together with $s_n \geq 1$ yields that $s_n < \dfrac{1+\sqrt{5}}{2}$ for $n \geq 1$.
Therefore, by the monotone convergence theorem that s_n converges as $n \to \infty$. Let its limit be s. Hence, $s_n \to s$ and $s_{n+1} \to s$ as $n \to \infty$. Setting $n \to \infty$ on both sides of $s_{n+1} = 1 + \dfrac{s_n}{1+s_n}$ yields $s = 1 + \dfrac{s}{1+s}$, which gives $s = \dfrac{1+\sqrt{5}}{2}$.

V. Let $s_n = \dfrac{\sin 1}{1^2} + \dfrac{\sin 2}{2^2} + \cdots + \dfrac{\sin n}{n^2}$ for $n \in \mathbb{N}$. Show that (s_n) is Cauchy.

Proof. Given an $\epsilon > 0$, let $N = \dfrac{1}{\epsilon}$. Then, for $m > n > N$,

$$\begin{aligned}
|s_m - s_n| &= |\sum_{k=n+1}^{m} \frac{\sin k}{k^2}| \leq \sum_{k=n+1}^{m} \frac{1}{k^2} \leq \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\
&= \sum_{k=n+1}^{m} (\frac{1}{k-1} - \frac{1}{k}) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon.
\end{aligned}$$

This shows that (s_n) is a Cauchy sequence.