

Problem 19.9. Let (s_n) be a bounded sequence and let S denote the set of subsequential limits of (s_n) . Prove that S is closed.

Proof. let S' be the set of accumulation points of S . In order to show that S is closed, it suffices to show $S' \subset S$. Let $t \in S'$. We shall show that $t \in S$. Since $t \in S'$, there is a sequence (t_m) with $t_m \in S$ such that $\lim_{m \rightarrow \infty} t_m = t$. Since each $t_m \in S$, there are subsequences $(s_{n_k^{(m)}})$ such that $\lim_{k \rightarrow \infty} s_{n_k^{(m)}} = t_m$ for each $m = 1, 2, \dots$, i. e.,

$$\begin{aligned} s_{n_1^{(1)}}, s_{n_2^{(1)}}, s_{n_3^{(1)}}, \dots, s_{n_k^{(1)}}, \dots &\rightarrow t_1, \\ s_{n_1^{(2)}}, s_{n_2^{(2)}}, s_{n_3^{(2)}}, \dots, s_{n_k^{(2)}}, \dots &\rightarrow t_2, \\ s_{n_1^{(3)}}, s_{n_2^{(3)}}, s_{n_3^{(3)}}, \dots, s_{n_k^{(3)}}, \dots &\rightarrow t_3, \\ \dots & \\ s_{n_1^{(m)}}, s_{n_2^{(m)}}, s_{n_3^{(m)}}, \dots, s_{n_k^{(m)}}, \dots &\rightarrow t_m, \\ \dots & \end{aligned}$$

Hence,

- there is $n_1 := n_{k_1}^{(1)}$ such that $|s_{n_1} - t_1| = |s_{n_{k_1}^{(1)}} - t_1| < 1$;
- there is $n_2 := n_{k_2}^{(2)}$ with $n_2 > n_1$ such that $|s_{n_2} - t_2| = |s_{n_{k_2}^{(2)}} - t_2| < 1/2$;
- there is $n_3 := n_{k_3}^{(3)}$ with $n_3 > n_2$ such that $|s_{n_3} - t_3| = |s_{n_{k_3}^{(3)}} - t_3| < 1/3$;
- \dots ;
- there is $n_m := n_{k_m}^{(m)}$ with $n_m > n_{m-1}$ such that $|s_{n_m} - t_m| = |s_{n_{k_m}^{(m)}} - t_m| < 1/m$;
- \dots .

We show that the subsequence $s_{n_m} \rightarrow t$ as $m \rightarrow \infty$. This follows directly from the following inequalities:

$$|s_{n_m} - t| \leq |s_{n_m} - t_m| + |t_m - t| \leq \frac{1}{m} + |t_m - t|.$$

Therefore, $t \in S$. This implies that $S' \subset S$. This shows that S is closed. □

Problem 19.11. Let (s_n) and (t_n) be bounded sequences. Show that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

Proof. Let A be the set of subsequential limits of $(s_n + t_n)$. Let $a \in A$. Then there is a subsequence $(s_{n_k} + t_{n_k})$ of $(s_n + t_n)$ such that $\lim_{k \rightarrow \infty} (s_{n_k} + t_{n_k}) = a$.

Since (s_{n_k}) is bounded, it follows from the B-W Theorem that (s_{n_k}) has a convergent subsequence, say $(s_{n_{k_i}})$, such that $s_{n_{k_i}} \rightarrow s$ as $i \rightarrow \infty$ for some $s \in \mathbb{R}$. A similar argument yields that there is a subsequence $(t_{n_{k_{i_j}}})$ such that $t_{n_{k_{i_j}}} \rightarrow t$ as $j \rightarrow \infty$ for some $t \in \mathbb{R}$. Hence,

$$s_{n_{k_{i_j}}} + t_{n_{k_{i_j}}} \rightarrow s + t \quad \text{as } j \rightarrow \infty.$$

Note that $(s_{n_{k_{i_j}}} + t_{n_{k_{i_j}}})$ is a subsequence of $(s_n + t_n)$. Hence, $\lim_{j \rightarrow \infty} (s_{n_{k_{i_j}}} + t_{n_{k_{i_j}}}) = \lim_{n \rightarrow \infty} (s_n + t_n) = a$. By the uniqueness of the limit it follows that $s + t = a$. Therefore, $a \leq \limsup s_n + \limsup t_n$. Thus, $\limsup s_n + \limsup t_n$ is an upper bound for A which yields

$$\limsup(s_n + t_n) = \sup A \leq \limsup s_n + \limsup t_n,$$

whence completing the proof. \square

Problem 19.14. Let (s_n) be a bounded sequence. Prove that $\limsup s_n = \lim_{n \rightarrow \infty} t_n$, where $t_n := \sup\{s_m : m > n\}$ for $n \in \mathbb{N}$.

Proof. Since (s_n) is bounded, it follows that t_n is well defined. We observe that t_n is decreasing and $t_n \geq s_m$ for all $m > n$, so that $t_n \geq \inf\{s_m : m \in \mathbb{N}\} > -\infty$. Hence by the Monotone Convergence Theorem, $t := \lim_{n \rightarrow \infty} t_n$ exists. The problem reduces to show that $\limsup s_n = t$. That is to show that t is the largest subsequential limit of (s_n) . We shall confirm this by the following two steps.

Step 1. Show that t is a subsequential limit of (s_n) .

- By the definition of t_1 , there exists $n_1 > 1$ such that $t_1 - 1 < s_{n_1} \leq t_1$;
- By the definition of t_{n_1} , there exists $n_2 > n_1$ such that $t_{n_1} - 1/2 < s_{n_2} \leq t_{n_1}$;
- By the definition of t_{n_2} , there exists $n_3 > n_2$ such that $t_{n_2} - 1/2 < s_{n_3} \leq t_{n_2}$;
- \dots ;
- By the definition of $t_{n_{k-1}}$, there exists $n_k > n_{k-1}$ such that $t_{n_{k-1}} - 1/k < s_{n_k} \leq t_{n_{k-1}}$;
- \dots .

Hence s_{n_k} is a subsequence of (s_n) . Since

$$t_{n_{k-1}} - 1/k < s_{n_k} \leq t_{n_{k-1}},$$

and $\lim_{k \rightarrow \infty} t_{n_k} = \lim_{n \rightarrow \infty} t_n = t$, it follows from the Squeeze Theorem that $\lim_{k \rightarrow \infty} s_{n_k} = t$. This shows that t is a subsequential limit of (s_n) .

Step 2. Show that if s is a subsequential limit of (s_n) , then $s \leq t$.

Let s be a subsequential limit of (s_n) . Then there is a subsequence (s_{n_k}) of (s_n) such that $\lim_{k \rightarrow \infty} s_{n_k} = s$. Note that $s_{n_k} \leq t_{(n_k-1)}$ for each $k \in \mathbb{N}$. We have

$$s = \lim_{k \rightarrow \infty} s_{n_k} \leq \lim_{k \rightarrow \infty} t_{(n_k-1)} = \lim_{n \rightarrow \infty} t_n = t.$$

\square

Supplementary Problem. Assume that $\lim_{n \rightarrow \infty} s_n = s$ with $s \in \mathbb{R}$. Show that

$$\lim_{n \rightarrow \infty} \frac{s_1 + s_2 + \dots + s_n}{n} = s.$$

Proof. First note that there is a $M > 0$ such that $|s_n - s| \leq M$ for $n \in \mathbb{N}$. Given an $\epsilon > 0$, there exists an integer $N_1 > 0$ such that $|s_n - s| < \epsilon/2$ for $n > N_1$. Note that for $n > N_1$, we have

$$\begin{aligned} \left| \frac{s_1 + s_2 + \dots + s_n}{n} - s \right| &\leq \frac{|s_1 - s| + |s_2 - s| + \dots + |s_{N_1} - s|}{n} + \frac{|s_{N_1+1} - s| + \dots + |s_n - s|}{n} \\ &\leq \frac{N_1 M}{n} + \frac{(n - N_1)\epsilon}{2n} \leq \frac{N_1 M}{n} + \frac{\epsilon}{2}. \end{aligned} \tag{0.1}$$

Below are the two arguments to finish the proof.

Method 1 (by the definition of the limit). Let $N = \max\{N_1, 2N_1M/\epsilon\}$. Then for $n > N$, we have $N_1M/n < \epsilon/2$ and so by (0.1),

$$\left| \frac{s_1 + s_2 + \cdots + s_n}{n} - s \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Method 2 (by the property of limit superior). Taking limit superior on both side of (0.1), we have

$$\limsup_{n \rightarrow \infty} \left| \frac{s_1 + s_2 + \cdots + s_n}{n} - s \right| \leq \limsup_{n \rightarrow \infty} \left(\frac{N_1M}{n} + \frac{\epsilon}{2} \right) = \frac{\epsilon}{2}.$$

Since ϵ is an arbitrarily positive number, it follows that

$$\limsup_{n \rightarrow \infty} \left| \frac{s_1 + s_2 + \cdots + s_n}{n} - s \right| = 0$$

which implies that

$$\lim_{n \rightarrow \infty} \left| \frac{s_1 + s_2 + \cdots + s_n}{n} - s \right| = 0,$$

and so

$$\lim_{n \rightarrow \infty} \left(\frac{s_1 + s_2 + \cdots + s_n}{n} - s \right) = 0,$$

and so

$$\lim_{n \rightarrow \infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = s.$$

□