Summary on variation of parameters

Consider the nonhomogeneous second-order linear equation

$$a(x)y'' + b(x)y' + c(x)y = g(x) \qquad (a(x) \neq 0). \tag{1}$$

The following are the steps to use the variation of parameters to find a particular solution of (1).

Step 1. Find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the corresponding homogeneous equation of (1):

$$a(x)y'' + b(x)y' + c(x)y = 0.$$

Step 2. Let $y_p(x) = y_1(x)u(x) + y_2(x)v(x)$ be a particular solution of (1). Then $u'(x)$ and $v'(x)$ satisfies

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{g(x)}{a(x)} \end{pmatrix};$$

that is,

$$\begin{cases} y_1(x)u'(x) + y_2(x)v'(x) = 0, \\ y_1'(x)u'(x) + y_2'(x)v'(x) = \frac{g(x)}{a(x)}. \end{cases}$$

Then solve for $u'(x)$ (or $v'(x)$) from the first equations in terms of $v'(x)$ (or $u'(x)$) and then substitute your solution into the second equation to solve $v'(x)$ and then $u'(x)$. Integrate your results for $u'(x)$ and $v'(x)$ and take the arbitrary integral constants to be zero to get functions $u(x)$ and $v(x)$. Finally, set $y_p(x) = y_1(x)u(x)+y_2(x)v(x)$. Indeed, we find that

$$u(x) = -\int \frac{y_2(x)g(x)}{a(x)W(x)}\,dx, \qquad v(x) = \int \frac{y_1(x)g(x)}{a(x)W(x)}\,dx,$$

so the general solution of (1) is given by the variation of parameters formula

$$y(x) = c_1y_1(x) + c_2y_2(x) - y_1(x)\int \frac{y_2(x)g(x)}{a(x)W(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{a(x)W(x)}\,dx.$$