Summary on two special types of 2nd-order equations

Type 1 (Equation not containing the unknown function y). The general form of this type of the equation is

$$F(x, y', y'') = 0. \tag{1}$$

Need two steps to solve (1).

- Let $v = y'$. Reduce (1) into the first order equation for $v = v(x)$:

  $$F(x, v, v') = 0, \tag{2}$$

  and then solve (2) for v (via the methods studied in Chapters 1-10.)

- Let $v = v(x, c_1)$ be the general solution of (2) obtained above. Now solve the directly integrable equation

  $$y' = v(x, c_1)$$

  to get the general solution $y = \int v(x, c_1)\, dx + c_2$ for (1).

Type 2 (Equation not containing the independent variable x). The general form of this type of the equation is

$$F(y, y', y'') = 0. \tag{3}$$

Need two steps to solve (3).

- Let $v = y'$. Regard y as the independent variable for v to get $\frac{dv}{dy} = \frac{dy'}{dx} / \frac{dy}{dx} = y''/y' = y''/v$ so that $y'' = v\frac{dv}{dy}$. Then reduce (3) into the first order equation for $v = v(y)$(with y as the independent variable):

  $$F(y, v, v\frac{dv}{dy}) = 0, \tag{4}$$

  and then solve (4) for $v = v(y)$ (via the methods studied in Chapters 1-10.)

- Let $v = v(y, c_1)$ be the general solution of (4) obtained above. Now solve the separable equation for $y = y(x)$ (with x as the independent variable):

  $$\frac{dy}{dx} = v(y, c_1)$$

  to get the general solution $y = y(x, c_1, c_2)$ defined (implicitly) by the equation

  $$\int \frac{1}{v(y, c_1)}\, dy = x + c_2$$

  for (3).