

Summary on Reduction of Order

Homogenous Equation. Consider

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad (x \in \mathcal{I}) \quad (1)$$

Assume that $y = y_1(x)$ is a known solution of (1). Let $y = y_1(x)u$ be an arbitrary solution of (1). Then u satisfies

$$[a_0(x)y_1(x)]u'' + [2a_0(x)y_1'(x) + a_1(x)y_1(x)]u' = 0.$$

This is the first (special) type of 2nd-order equation, which can be solved with a general solution

$$u(x) := U(x, c_1) + c_2.$$

Let $c_1 = 1$ and $c_2 = 0$ to get a special solution $u(x) = U(x, 1)$. Then

$$y_2(x) := y_1(x)U(x, 1)$$

gives a solution of (1), which is linearly independent of $y_1(x)$.

Nonhomogenous Equation. Consider

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x) \quad (x \in \mathcal{I}) \quad (2)$$

Assume that $y = y_1(x)$ is a known solution of (1). Let $y = y_1(x)u$ be an arbitrary solution of (2). Then u satisfies

$$[a_0(x)y_1(x)]u'' + [2a_0(x)y_1'(x) + a_1(x)y_1(x)]u' = g(x).$$

This is the first (special) type of 2nd-order equation, which can be solved with a general solution

$$u(x) = U(x, c_1) + c_2.$$

Then

$$y(x) := y_1(x) \cdot [U(x, c_1) + c_2]$$

gives a general solution of (2).