Summary on Euler's Numerical Method

Let $y = y(x)$ be the solution of the IVP:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0. \tag{1}$$

- Given a set of finite mesh points $\{x_0 < x_1 < \cdots < x_N\}$, Euler's numerical method defines an approximate value y_k to each $y(x_k)$.
- Start from the point (x_0, y_0), define sequentially (x_{k+1}, y_{k+1}) for $k = 0, 1, \cdots, N-1$ to be the right endpoint of slope line of (1) over the interval $[x_k, x_{k+1}]$ through (x_k, y_k), so that

  $$y_{k+1} = y_k + \Delta x_k \cdot f(x_k, y_k) \qquad k = 0, 1, \cdots, N-1.$$

  where $\Delta x_k = x_{k+1} - x_k$.
- In general, x_k are defined equally spaced with step size $h > 0$ so that

  $$x_{k+1} = x_k + h \qquad k = 0, 1, \cdots, N-1. \tag{2}$$

  Hence,

  $$y_{k+1} = y_k + h \cdot f(x_k, y_k) \qquad k = 0, 1, \cdots, N-1. \tag{3}$$

  (2) and (3) are called the Euler's iterative formulas.
- In general, y_k (except $k = 0$) may not be equal to $y(x_k)$. It can be shown that the error $|y_k - y(x_k)|$ is proportional to the step size h. Thus, doubling the number of mesh points N will decrease the error by 1/2.