Summary on the 2nd-order Euler equation

Consider the homogeneous second-order Euler equation

$$ax^2y'' + bxy' + cy = 0 \qquad (x \neq 0) \tag{1}$$

which has solutions of the form $y(x) = x^r$ where r is a constant satisfying the indicial equation

$$ar(r-1) + br + c = ar^2 + (b-a)r + c = 0. \tag{2}$$

The general solution of (1) is given as follows.

- If (2) has two distinct **real** roots r_1 and r_2, then the general solution of (1) is given by

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}.$$

- If (2) has a **repeated** roots $r_1 = r_2$, then the general solution of (1) is given by

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_1}\ln|x| = |x|^{r_1}(c_1 + c_2\ln|x|)$$

- If (2) has two distinct **complex conjugate** roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then the general solution of (1) is given by

$$\begin{aligned} y(x) &= c_1|x|^{\alpha}\cos(\beta\ln|x|) + c_2|x|^{\alpha}\sin(\beta\ln|x|) \\ &= |x|^{\alpha}[c_1\cos(\beta\ln|x|) + c_2\sin(\beta\ln|x|)]. \end{aligned}$$