Basic theorems on linear $2nd$-order equations

Theorem A (Homogenous equation). Consider

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \qquad (x \in \mathcal{I}) \tag{1}$$

where $\mathcal{I}$ is an interval, $a_i(x)$ are continuous for $x \in \mathcal{I}$ $(i = 0, 1, 2)$, and $a_0(x) \neq 0$ for $x \in \mathcal{I}$. The following holds.

(i) (**Superposition principle**.) If y_1 and y_2 are the solutions of (1), then, given any constants c_1 and c_2, the linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{2}$$

is also a solution of (1).

(ii) (**General solution**.) If y_1 and y_2 are linearly independent, then (2) gives a general solution of (1). That is, given any constants A, B and $x_0 \in \mathcal{I}$, there is a unique pair of constants c_1 and c_2 that solve

$$c_1 y_1(x_0) + c_2 y_2(x_0) = A, \qquad c_1 y_1'(x_0) + c_2 y_2'(x_0) = B,$$

so that $y = y(x)$ defined by (2) with this pair c_1 and c_2 solves the initial value problem

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \qquad y(x_0) = A, \quad y'(x_0) = B.$$

Theorem B (Nonhomogenous equation). Consider

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x) \qquad (x \in \mathcal{I}) \tag{3}$$

where $\mathcal{I}$ is an interval, $a_i(x)$ and $g(x)$ are continuous for $x \in \mathcal{I}$ $(i = 0, 1, 2)$, $a_0(x) \neq 0$ for $x \in \mathcal{I}$. The following holds.

(i) (**Superposition principle**.) If y_h is a solution of **(1)** and y_p is a solution of **(3)**, then

$$y = y_h + y_p$$

is solution of **(3)**. The difference of any two solutions of **(3)** is a solution of **(1)**.

(ii) (**General solution**.) If y_1 and y_2 are linearly independent solutions of **(1)** and y_p is a (particular) solution of **(3)**, then

$$y = c_1 y_1 + c_2 y_2 + y_p \tag{4}$$

is a general solution of **(3)**. That is, given any constants A, B and $x_0 \in \mathcal{I}$, there is a unique pair of constants c_1 and c_2 that solve

$$c_1 y_1(x_0) + c_2 y_2(x_0) + y_p(x_0) = A, \qquad c_1 y_1'(x_0) + c_2 y_2'(x_0) + y_p'(x_0) = B,$$

so that $y = y(x)$ defined by **(4)** with this pair c_1 and c_2 solves the initial value problem

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x), \qquad y(x_0) = A, \quad y'(x_0) = B.$$