Solutions to Exam #2 **Name: Shangbing Ai**

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1. Given $\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + c$, find $\int \frac{1}{(2+2x+x^2)^2}\,dx$.
(Hint: Write $2+2x+x^2 = b+(a+x)^2$ for some a and b and then use a substitution.)

Solution. Since $2+2x+x^2 = 1+(1+x)^2$, we let $u = 1+x$ to yield $du = dx$ and

$$\begin{aligned}\int \frac{1}{(2+2x+x^2)^2}\,dx &= \int \frac{1}{[1+(1+x)^2]^2}\,dx = \int \frac{1}{(1+u^2)^2}\,du \\ &= \frac{1}{2}\arctan u + \frac{u}{2(1+u^2)} + c = \frac{1}{2}\arctan(1+x) + \frac{1+x}{2[1+(1+x)^2]} + c.\end{aligned}$$

2. (i) Find A, B, C so that $\frac{2x^2+x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$.

Solution. Multiplying both sides of the given equation by $(x-1)(x^2+1)$ to get

$$\begin{aligned}2x^2+x-1 &= A(x^2+1)+(Bx+C)(x-1) \\ &= Ax^2+A+Bx^2+Cx-Bx-C = (A+B)x^2+(C-B)x+A-C.\end{aligned}$$

By equating the coefficients of the same powers of x we get

$$A+B=2, \qquad C-B=1, \qquad A-C=-1.$$

Solving this linear system to get $A=1$, $b=1$ and $C=2$. Thus,

$$\frac{2x^2+x-1}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{x+2}{x^2+1}.$$

(ii) Evaluate $\int \frac{2x^2+x-1}{(x-1)(x^2+1)}\,dx$.

Solution. From the result of (i) we have

$$\begin{aligned}\int \frac{2x^2+x-1}{(x-1)(x^2+1)}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx \\ &= \ln|x-1| + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + 2\arctan x \\ &= \ln|x-1| + \frac{1}{2}\ln(x^2+1) + 2\arctan x + c.\end{aligned}$$

3. Determine $\int_2^6 \frac{1}{\sqrt{y-2}}\,dy$ is convergent or divergent and find its value if it is convergent.

Solution. We have

$$\int_2^6 \frac{1}{\sqrt{y-2}}\,dy = \lim_{a\to 2^+}\int_a^6 \frac{1}{\sqrt{y-2}}\,dy = \lim_{a\to 2^+} 2\sqrt{y-2}\Big|_a^6 = \lim_{a\to 2^+}\Big[4-2\sqrt{a-2}\Big] = 4.$$

Hence, the given integral is convergent to 4.

4. Find the length of the curve $y = x^2 - \frac{1}{8}\ln x$, $1 \le x \le 4$

Solution. First we have $y' = 2x - 1/8x$ and so

$$1+(y')^2 = 1+4x^2 - \frac{1}{2} + \frac{1}{64x^2} = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \Big(2x + \frac{1}{8x}\Big)^2.$$

Hence, the length of the curve

$$L = \int_1^4 \sqrt{1+(y')^2}\,dx = \int_1^4 \Big(2x + \frac{1}{8x}\Big)\,dx = \Big(x^2 + \frac{1}{8}\ln x\Big)\Big|_1^4$$
$$= 16 + \frac{1}{8}\ln 4 - 1 = 15 - \frac{1}{4}\ln 2.$$

5. Find the centroid of the region as shown in the figure:

Solution. The area of the region A is the sum of areas of the half unit circle and the square of sides 2 and so $A = \pi/2 + 4$. Since the region is symmetric with the y axis we expect the x-coordinate $\bar{x}$ of the centroid be zero, i.e., $\bar{x} = 0$. Indeed by the general formulas

$$\bar{x} = \frac{1}{A}\int_{-1}^1 x[f(x)-g(x)]\,dx, \qquad \bar{y} = \frac{1}{2A}\int_{-1}^1 [f^2(x)-g^2(x)]\,dx,$$

we have, with the top curve $f(x) = \sqrt{1-x^2}$ and the bottom curve $g(x) = -2$,

$$\bar{x} = \frac{1}{A}\int_{-1}^1 x(\sqrt{1-x^2}+2)\,dx = 0,$$

since the integrand is odd and upper and lower limits are symmetric;

$$\bar{y} = \frac{1}{2A}\int_{-1}^1 (1-x^2-4)\,dx = \frac{1}{2A}\left(-\frac{1}{3}x^3 - 3x\right)\Big|_{-1}^1 = \frac{1}{A}\Big(-\frac{1}{3}-3\Big)$$
$$= -\frac{10}{3A} = -\frac{20}{3(\pi+8)}.$$

6 (**Bouns**). (i) Find the solution to the initial-value problem:

$$\frac{dy}{dx} = \frac{1+x}{xy} \quad (x>0), \quad y(1) = -4.$$

Solution. First to find the general solutions for the equation by the separable method. Informally, multiplying both sides of the equation by $y\,dx$ to get $y\,dy = [(1+x)/x]\,dx$, then integrate to give

$$\frac{1}{2}y^2 = \int \frac{1+x}{x}\,dx = \int \left(\frac{1}{x}+1\right)dx = \ln|x| + x + c.$$

Evaluate the above equation at $x = 1$ and use the initial value $y(1) = -4$ to get $8 = 1+c$ and so $c = 7$. Therefore, the desired solution $y = y(x)$ is defined implicitly by $\frac{1}{2}y^2 = \ln|x|+x+7$.

(ii) Use a trigonometric substitution to evaluate $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$.

Solution. Let $x = \tan\theta$. we have $dx = \sec^2\theta\,dx$ and $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$, and so

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c.$$

It remains to substitute the variable θ by the original variable x. Note that $\theta = \arctan x$ and

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\tan\theta\cos^2\theta = \frac{2\tan\theta}{\sec^2\theta} = \frac{2\tan\theta}{1+\tan^2\theta}.$$

(This is also a double angle formula for $\sin 2\theta$ in terms of $\tan\theta$.) Therefore, we have

$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + c.$$