Write your full name and answer the questions on the paper provided. Work neatly and carefully.

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1)[30pts]
(i) Let A and $A+E$ be nonsingular,

$$A\mathbf{x} = \mathbf{b}, \qquad (A+E)\mathbf{x}_c = \mathbf{b}$$

where b is a non-zero vector. Prove directly that

$$\frac{\|\mathbf{x}-\mathbf{x}_c\|_\infty}{\|\mathbf{x}_c\|_\infty} \le \kappa_\infty(A)\frac{\|\mathbf{E}\|_\infty}{\|A\|_\infty}.$$

Interpret this result.
(ii) Next, let

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}, \qquad E = \begin{pmatrix} 0.9 & 0.2 \\ 0.3 & 0.4 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Use the estimate obtained above to find a lower bound for $\kappa_\infty(A)$.
(iii) Check this estimate by computing $\kappa_\infty(A)$ directly.

2)[45pts]
(i) Let $A \in \mathbb{R}^{n\times m}$ with $n \ge m$.
(a) Show that $\|A^TA\|_2 = \|A\|_2^2$
(b) Show that $\kappa_2(A^TA) = \kappa_2(A)^2$. Interpret this result in the context of solving ill-conditioned least squares problems by normal equations approach.
(ii) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 2 & 4 \end{pmatrix}.$$

Determine the singular value decomposition (SVD) of A in the form $A = U\Sigma V^T$.
(iii) Use the above SVD to find the minimum norm least squares solution to

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 5 \end{pmatrix}.$$

3)[25pts] Given the data $(0,1), (2,4), (5,6)$. Use a QR factorization technique to find the best least squares fit by a linear function.